# Power Method

- Given a matrix $A \in C^{n\times n}$, the power method is an iteration procedure defined as follows:
  - ◇ Begin with an arbitrary $x^{(0)} \in C^n$.
  - ◇ Repeatedly define the sequence $\{x^{(k)}\}$ by
$$w^{(k)} := Ax^{(k-1)}$$
$$x^{(k)} := \frac{w^{(k)}}{\|w^{(k)}\|_\infty}$$
    for $k = 1, 2, \ldots$ until convergence.
- The normalization is for the purpose of avoiding overflow or underflow.
  - ◇ Any norm can be used for the normalization. The sup-norm is particularly convenient.
  - ◇ The normalization needs not be done at every step because
$$\begin{aligned} x^{(k)} &= \frac{Ax^{(k-1)}}{\|Ax^{(k-1)}\|_\infty} &= \frac{A\left(\frac{w^{(k-1)}}{\|w^{(k-1)}\|_\infty}\right)}{\|A\left(\frac{w^{(k-1)}}{\|w^{(k-1)}\|_\infty}\right)\|_\infty} \\ &= \frac{A^2x^{(k-2)}}{\|A^2x^{(k-2)}\|_\infty} &= \frac{A^kx^{(0)}}{\|A^kx^{(0)}\|_\infty}. \end{aligned}$$
- Assume $A$ is diagonalizable with eigenvalues $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$ and corresponding eigenvectors $x_1, \ldots x_n$.
  - ◇ Write $x^{(0)} = \sum_{i=1}^n \alpha_i x_i$. (This is possible because $A$ is assumed to be diagonalizable.)
  - ◇ Note that
$$\begin{aligned} Ax^{(0)} &= \sum_{i=1}^n \alpha_i\lambda_i x_i \\ A^kx^{(0)} &= \sum_{i=1}^n \alpha_i\lambda_i^k x_i = \lambda_1^k\left(\alpha_1 x_1 + \sum_{i=2}^n \alpha_i\left(\frac{\lambda_i}{\lambda_1}\right)^k x_i\right). \end{aligned}$$

- ◇ Assume $\alpha_1 \neq 0$. (This is guaranteed if $x^{(0)}$ is selected randomly.)
- ◇ Note that as $k \to \infty$, the vector $A^k x^{(0)}$ *behaves like* $\alpha_1 \lambda_1^k x_1$ in the sense that contribution from $x_2, \ldots x_n$ becomes less and less significant.
- ◇ Normalization makes $x^{(k)} \to \frac{\alpha_1 \lambda_1^k}{|\alpha_1 \lambda_1^k|} \frac{x_1}{\|x_1\|_\infty}$.
- ◇ The sequence $\{x^{(k)}\}$ converges to an eigenvector associated with the eigenvalue $\lambda_1$.
- ◇ Also, $w^{(k+1)} = Ax^{(k)} \to \lambda_1 x^{(k)}$. So $\frac{w^{(k+1)})_j}{x_j^{(k)}} \to \lambda_1$.

- The rate of convergence of power method depends on the ratio $\frac{\lambda_2}{\lambda_1}$.