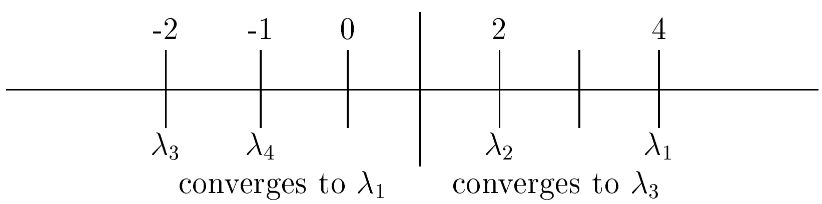
Inverse Power Method

- The eigenvalues of the matrix $A - bI$ for a scalar b are $\lambda_i - b$ if and only if λ_i are eigenvalues of A.
  - ◇ The idea can be used to work on the matrix $A - bI$ instead of A with the hope that the ratio of the first two dominant eigenvalues $\lambda_i - b$ will become smaller. This is called a *shifted power method.*
  - ◇ The application is very limited
  - ◇ Assume all eigenvalues are real and are distributed as follows:



  - ▷ with all choices of b, the shifted power method will converge to either λ_1 or λ_3, but not any other eigenvalues.

- Suppose A is nonsingular, then A^{-1} has eigenvalues $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$. We may apply the power method to A^{-1}. This is called the *inverse power method.*
  - ◇ Begin with an arbitrary $x^{(0)} \in C^n$.
  - ◇ Repeatedly define the sequence $\{x^{(k)}\}$ by

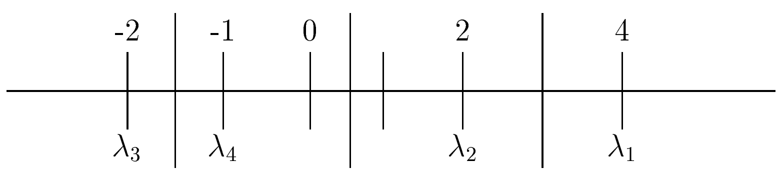
$$Aw^{(k)} := x^{(k-1)}$$
$$x^{(k)} := \frac{w^{(k)}}{\|w^{(k)}\|_\infty}$$

for $k = 1, 2, \ldots$ until convergence.

- Using the same analysis before, if we assume $|\lambda_1| \geq \ldots \geq |\lambda_{n-1}| > |\lambda_n|$, then $\{x^{(k)}\}$ produced by the inverse power method converges to an eigenvector associated with λ_n at the rate depending on $|\frac{\lambda_n}{\lambda_{n-1}}|$.
- If we also bring the idea of shift, then we obtain the *shifted inverse power method.*
  - ◇ Begin with an arbitrary $x^{(0)} \in C^n$.
  - ◇ Repeatedly define the sequence $\{x^{(k)}\}$ by

$$(A - bI)w^{(k)} := x^{(k-1)}$$
$$x^{(k)} := \frac{w^{(k)}}{\|w^{(k)}\|_\infty}$$

    for $k = 1, 2, \ldots$ until convergence.
- The eigenvalues of $(A - bI)^{-1}$ are $\frac{1}{\lambda_1 - b}, \ldots, \frac{1}{\lambda_n - b}$.
  - ◇ Whenever b is chosen close enough to the eigenvalue λ_i, the sequence $\{x^{(k)}\}$ by the shifted inverse power method converges to an eigenvector associate with the eigenvalue $\frac{1}{\lambda_i - b}$.
  - ◇ Suppose eigenvalues are distributed as follows:



    - ▷ The longer vertical bars separates the regions of b by which the shifted inverse power method will be able to find, respectively, different eigenvalues.
- There are several strategies in selecting the shift b.
  - ◇ If some estimate of λ_i has been found, we may use it for b.
  - ◇ Generate $b^{(0)}$ randomly. Then define

$$b^{(k+1)} := \frac{(w^{(k)})^* A w^{(k)}}{(w^{(k)})^* w^{(k)}} \tag{0.1}$$

    successively. This is known as the *Rayleigh quotient iteration.*