Convergence of QR Algorithm

- The QR algorithm can be considered as a numerical way to realize the Schur decomposition theorem that every square matrix A can be transformed by unitary similarity transformation into an upper triangular matrix, i.e., there exist an unitary matrix $Q \in C^{n \times n}$ such that

$$A = QTQ^*, \tag{0.1}$$

  where T is upper triangular.

- Suppose the matrix $A \in \mathbf{R^{n \times n}}$ has distinct real eigenvalues. Then the sequence $\{A_k\}$ generated by the QR algorithm converges to an upper triangular matrix.
  - ◇ The real arithmetic used by QR algorithm *cannot* produce complex values.
  - ◇ A real matrix could have complex conjugate pairs of eigenvalues. In that case, the QR algorithm will not converge in the usual sense.
  - ◇ The sequence $\{A_k\}$ will have a *pseudo-convergence* behavior, i.e., it will appear that the sequence $\{A_k\}$ converges to a block upper triangular matrix of sizes 1×1 or 2×2. Elements in the rows corresponding to the 2×2 block will never converge to fixed values. Instead, these elements constitute a periodic curve in the complex plane.
  - ◇ Regardless of the non-convergence, eigenvalues of the 2×2 block are readily available.