Some Classical Splittings

- The choice of S gives rise to different iterative schemes.
- Let the matrix A be split as

$$A = D - L - U \tag{0.1}$$

where D, $-L$ and $-U$ are, respectively, the diagonal, the strictly lower triangular and the strictly upper triangular matrices of A.

- The Jacobi Method:
  - ◇ Splitting: $S := D$, $T = L + U$.
  - ◇ Matrix form: $Dx^{new} = (L + U)x^{old} + b$.
  - ◇ Component form: $x_i^{new} = \frac{1}{a_{ii}}(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{old} - \sum_{j=i+1}^{n} a_{ij}x_j^{old})$.
- The Gauss-Seidel Method:
  - ◇ Splitting: $S := D - L$, $T = U$.
  - ◇ Matrix form: $(D - L)x^{new} = Ux^{old} + b$.
  - ◇ Component form: $x_i^{new} = \frac{1}{a_{ii}}(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{new} - \sum_{j=i+1}^{n} a_{ij}x_j^{old})$.
- The SOR Method:
  - ◇ Splitting: $S := \frac{1}{\omega}D - L$, $T = (\frac{1}{\omega} - 1)D + U$.
  - ◇ Matrix form: $(D - \omega L)x^{new} = (1 - \omega)Dx^{old} + \omega Ux^{old} + \omega b$.
  - ◇ Component form: $x_i^{new} = (1 - \omega)x_i^{old} + \omega\hat{x}_i^{new}$ where
    - ▷ $\hat{x}_i^{new} = \frac{1}{a_{ii}}(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{new} - \sum_{j=i+1}^{n} a_{ij}x_j^{old})$.
- The geometric meaning of these methods can be seen from the iterates on the 2×2 problem:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$