Some Convergence Theorems

- The following three statements are equivalent:
  - ◇ $\lim_{n\to\infty} H^n = 0$.
  - ◇ $\lim_{n\to\infty} \|H^n\| = 0$ for *some* norm.
  - ◇ The spectral radius $\rho(H) < 1$.
- Suppose $x = Hx + d$ has a unique solution x^*. Then the sequence $\{x^{(k)}\}$ computed from the scheme

$$x^{new} = Hx^{old} + d \tag{0.1}$$

  with any starting point $x^{(0)}$ converges to x^* if and only $\rho(H) < 1$.
  - ◇ Observe that $x^{(k+1)} - x^* = H(x^{(k)} - x^*) = \ldots = H^{k+1}(x^{(0)} - x^*)$.
  - ◇ If $\rho(H) < 1$, then $\|x^{(k+1)} - x^*\| \leq \|H^{k+1}\|\|x^{(0)} - x^*\| \to 0$. Convergence follows from the above theorem.
  - ◇ Suppose $\|x^{(k)} - x^*\| \to 0$ for every $x^{(0)}$. Take $x^{(0)} = x^* + e_i$. Then $x^{(k)} - x^* = H^k e_i =$ The i-th column of H^k. Use the $\|\cdot\|_1$ norm, then we have $\|$The i-th column of $H^k\|_1 \to 0$ as $k \to \infty$. Since i is arbitrary, it follows that $\|H^k\|_1 \to 0$.
- A main task in the splitting $A = S - T$ is to make sure that the spectral radius of the iteration matrix $H = S^{-1}T$ is strictly less than 1.
- Both the Jacobi method and the Gauss-Seidel method converge if A is strictly diagonally dominant.
- If A is symmetric and positive definite, then the Gauss-Seidel method converges.
- (Ostrowski and Reich Theorem) Let A be real, symmetric and positive definite. Then the SOR method converges if and only if $0 < \omega < 2$.