Polynomial Acceleration Method

- Other than convergence, another main concern in any iterative method is its speed of convergence.
- One procedure to accelerate convergence involves the formation of a new vector sequence from linear combinations of the iterates obtained from the basic method.
- The idea:
  - ◇ Let $\{x^{(k)}\}$ be the sequence of iterates generated from the basic scheme
    $$x^{(k)} = Hx^{(k-1)} + d. \tag{0.1}$$
  - ◇ The error vector $e^{(k)} := x^{(k)} - x^*$ satisfies
    $$e^{(k)} = H^k e^{(0)}. \tag{0.2}$$
  - ◇ Consider a new vector sequence $\{u^{(k)}\}$ determined by the linear combination
    $$u^{(k)} := \sum_{i=0}^{k} \alpha_{k,i} x^{(i)}, k = 0, 1, \ldots \tag{0.3}$$
    - ▷ The real numbers $\alpha_{k,i}$ are required to satisfy the consistency condition
      $$\sum_{i=0}^{k} \alpha_{k,i} = 1, \quad k = 0, 1, \ldots \tag{0.4}$$
  - ◇ The new errors $\epsilon^{(k)} := u^{(k)} - x^*$ satisfy
    $$\begin{aligned} \epsilon^{(k)} &= \textstyle\sum_{i=0}^{k} \alpha_{k,i} x^{(i)} - x^* = \sum_{i=0}^{k} \alpha_{k,i} e^{(i)} \\ &= (\textstyle\sum_{i=0}^{k} \alpha_{k,i} H^i) e^{(0)} = (\sum_{i=0}^{k} \alpha_{k,i} H^i) \epsilon^{(0)} \\ &:= Q_k(H) \epsilon^{(0)}. \end{aligned}$$
    - ▷ The acceleration depends on the matrix polynomial
      $$Q_k(H) := \alpha_{k,0} I + \alpha_{k,1} H + \ldots + \alpha_{k,k} H^k.$$

- ◇ The idea is to choose the polynomials $\{Q_k\}$ so that $\{u^{(k)}\}$ converges to x^* faster than $\{x^{(k)}\}$.
- ◇ The polynomial can be chosen to fulfill one of two purposes:
  - ▷ To minimize $\|Q_k(H)\epsilon^{(0)}\|$.
  - ▷ To minimize the virtual spectral radius

$$\overline{\rho}(Q_k(H)) := \max_{m(H)\leq x\leq M(H)} |Q_k(x)|.$$

- Generally speaking, it requires a high arithmetic cost and a large amount of storage in using (0.3) to obtain $u^{(k)}$.

- Alternatively, we usually consider polynomials satisfying the three-term recurrence relation:

$$\begin{aligned} Q_0(x) &= 1 \\ Q_1(x) &= \gamma_1 x - \gamma_1 + 1 \\ Q_{k+1}(x) &= \rho_{k+1}(\gamma_{k+1}x + 1 - \gamma_{k+1})Q_k(x) + (1-\rho_{k+1})Q_{k-1}(x), \text{ for } k \geq 1 \end{aligned} \tag{0.5}$$

where $\gamma_1, \rho_2, \gamma_2, \ldots$ are real numbers to be determined.

- ◇ One of the main advantage of the above three-term recurrence relation is that the sequence $\{u^{(k)}\}$ can be generated from

$$\begin{aligned} u^{(1)} &= \gamma_1(Hu^{(0)} + d) + (1-\gamma_1)u^{(0)}, \\ u^{(k+1)} &= \rho_{k+1}\{\gamma_{k+1}(Hu^{(k)} + d) + (1-,\gamma_{k+1})u^{(k)} + (1-\rho_{k+1})u^{(k-1)}. \end{aligned} \tag{0.6}$$