Krylov Subspace Methods

- Recently it has be discovered that another class of iterative methods, based on the Krylov subspace, performs more competitively with the classical stationary iterative methods. Two basic methods in this class are most interesting:
  - ⋄ Conjugate Gradient Iteration for symmetric and definite matrices.
  - ⋄ GMRES (Generalized Minimum RESidual) for nonsymmetric matrices.
- Given a matrix A and an arbitrary vector x_0, define $r_0 = b - Ax_0$. The subspace
$$\mathcal{K}_k := \text{span}\left(r_0, Ar_0, \ldots, A^{k-1}r_0\right)$$
is called the k-th Krylov subspace associated with A and x_0.
- The general idea in this direction is to minimize some desired objective function over a sequence of subspaces that is expanding one dimension a time. Hopefully, the minimization involved can be handled efficiently.
  - ⋄ In the class of Krylov methods, the subspaces are taken to be $x_0 + \mathcal{K}_k$, $k = 1, \ldots, n$.
  - ⋄ Since the ultimate dimension is n, it is expected that Krylov methods should terminate in n steps.
- Objective functions:
  - ⋄ In the conjugate gradient method, the objective is to
$$\min_{w \in x_0 + \mathcal{K}_k} \|x^* - w\|_A \tag{0.1}$$
where $\|x\|_A := \sqrt{x^T A x}$.
  - ⋄ In the GMRES method, the objective is to
$$\min_{x \in x_0 + \mathcal{K}_k} \|b - Ax\|_2. \tag{0.2}$$