Lecture 2: Basic Language

Hilbert Space, Dirac Notation, and Dual Operators

Moody T. Chu

NC State University

Spring 2026

Lecture Overview

1. Complex Vector Spaces and Hilbert Space
2. Formal Axioms of the Inner Product
3. Dirac Notation: Kets
4. Linear Operators and the Adjoint
5. The Dual Space, Bras, and the Riesz Theorem
6. Orthonormality and the Gram-Schmidt Process
7. Outer Products and the Completeness Relation
8. Spectral Decomposition

The State Space: Complex Vector Spaces

Physical states inhabit a **complex vector space** $\mathcal{V}$ over the field $\mathbb{C}$.

- **Closure axioms:**
  - ◇ Under addition: $|u\rangle + |v\rangle \in \mathcal{V}$ for all $|u\rangle, |v\rangle \in \mathcal{V}$.
  - ◇ Under scalar multiplication: $\alpha |u\rangle \in \mathcal{V}$ for all $\alpha \in \mathbb{C}$.
- **Additional structure:** existence of a zero vector $|0\rangle$ and additive inverses.
- **Dimension in quantum computing:**
  - ◇ A single qubit lives in $\mathbb{C}^2$ (dimension 2).
  - ◇ A system of n qubits lives in $\mathbb{C}^{2^n}$ (dimension 2^n).
  - ◇ This exponential growth of dimension underlies the computational power of quantum systems.

Axioms of the Inner Product

Enriching $\mathcal{V}$ with an **inner product** $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ adds geometry:

- **Axiom 1 — Positivity:** $\langle v, v \rangle \geq 0$, with $\langle v, v \rangle = 0 \iff v = 0$.
- **Axiom 2 — Linearity in the second argument:** $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle$.
- **Axiom 3 — Conjugate symmetry:** $\langle u, v \rangle = \overline{\langle v, u \rangle}$.

Axioms 2 and 3 together imply **conjugate linearity in the first argument**:

$$\begin{aligned} \langle \alpha u + \beta v,\, w \rangle &= \overline{\langle w,\, \alpha u + \beta v \rangle} = \overline{\alpha \langle w, u \rangle + \beta \langle w, v \rangle} \\ &= \alpha^* \langle u, w \rangle + \beta^* \langle v, w \rangle. \end{aligned}$$

This asymmetry — linear in the second slot, conjugate-linear in the first — is essential for Dirac notation.

Hilbert Space

The inner product induces a **norm**: $\|v\| = \sqrt{\langle v, v\rangle} \geq 0$.

- A complex inner product space that is *complete* under this norm is called a **Hilbert space** $\mathscr{H}$.
  - ◇ *Complete* means every Cauchy sequence of vectors converges to a vector *within* $\mathscr{H}$.
  - ◇ In finite dimensions (our setting in quantum computing), every inner product space is automatically complete.
- **Physical interpretation of the inner product:**
  - ◇ The *norm* measures the "length" of a state vector; physical states are *unit* vectors ($\|v\| = 1$).
  - ◇ For a unit vector, $|\langle e_i, \psi\rangle|^2$ gives the *probability* of finding the system in state $|e_i\rangle$.
  - ◇ The *angle* between two vectors governs interference and distinguishability.

Dirac Notation: The Ket

In Dirac notation, a state vector in $\mathscr{H}$ is written as a **ket**: $|\psi\rangle$.

- **Column vector representation:** given a basis $\{|i\rangle\}$,

$$|\psi\rangle = \sum_i a_i |i\rangle \quad \longleftrightarrow \quad \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix}, \qquad a_i \in \mathbb{C}.$$

- **Normalization:** a physical (pure) state satisfies $\langle\psi|\psi\rangle = \| \, |\psi\rangle \, \|^2 = 1$.
- **Global phase:** $|\psi\rangle$ and $e^{i\theta} |\psi\rangle$ represent the *same* physical state for any $\theta \in \mathbb{R}$ — no measurement can distinguish them (as established in Lecture 1).

Notation convention: we write $\langle u|v\rangle$ for the inner product $\langle |u\rangle , |v\rangle\rangle$, and drop the outer parentheses for readability.

Linear Operators

An operator $A : \mathscr{H} \to \mathscr{K}$ is **linear** if for all scalars $\alpha_i \in \mathbb{C}$ and vectors $|\psi_i\rangle \in \mathscr{H}$:

$$A\left(\sum_i \alpha_i \,|\psi_i\rangle\right) = \sum_i \alpha_i \, A \,|\psi_i\rangle \,.$$

- A linear map $f : \mathscr{H} \to \mathbb{C}$ is called a **linear functional**.
- A linear map $T : \mathscr{H} \to \mathscr{H}$ is called a **linear transformation**.
- **Matrix representation:** in orthonormal bases $\{|e_i\rangle\}$ for $\mathscr{H}$ and $\{|f_j\rangle\}$ for $\mathscr{K}$, the (j, i) entry of A is

$$A_{ji} = \langle f_j \mid A \, e_i \rangle.$$

- **Operator expansion:** $A = \sum_{j,i} A_{ji} \, |f_j\rangle \langle e_i|$ (outer products — to be defined shortly).

The Adjoint Operator

Theorem: For every linear operator $A : \mathscr{H} \to \mathscr{K}$, there exists a *unique* operator $A^\dagger : \mathscr{K} \to \mathscr{H}$, called the **adjoint** of A, satisfying:

$$\langle Ah,\, k\rangle_{\mathscr{K}} = \langle h,\, A^\dagger k\rangle_{\mathscr{H}} \quad \text{for all } h \in \mathscr{H},\ k \in \mathscr{K}.$$

- In any orthonormal basis, $A^\dagger$ is represented by the *conjugate transpose* of A: $(A^\dagger)_{ij} = \overline{A_{ji}}$.
- **Key identities:** $(AB)^\dagger = B^\dagger A^\dagger, \quad (A^\dagger)^\dagger = A, \quad (\alpha A)^\dagger = \alpha^* A^\dagger$.
- Two important special cases:
  - ◇ $A = A^\dagger$: **Hermitian** (self-adjoint). Physical observables are Hermitian.
  - ◇ $U^\dagger U = I$: **Unitary**. Reversible quantum gates are unitary.
- How does one find $A^\dagger$ for a non-matrix operator?
- The adjoint concept extends to nonlinear operators (beyond our scope here).

The Dual Space and the Bra

- The **dual space** $\mathscr{H}^*$ is the set of all *linear functionals* $f : \mathscr{H} \to \mathbb{C}$.
- Each element of $\mathscr{H}^*$ is written as a **bra**: $\langle\psi|$.
- **Connection to the adjoint:** the bra $\langle\psi|$ is the adjoint of the ket $|\psi\rangle$,

$$\langle\psi| = |\psi\rangle^\dagger,$$

consistent with taking the conjugate transpose of the column vector:

$$|\psi\rangle = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix} \quad \longrightarrow \quad \langle\psi| = \left[a_1^*,\ \ldots,\ a_d^*\right].$$

- The bra $\langle\psi|$ *acts on* a ket $|\phi\rangle$ via the inner product:

$$\langle\psi|\left(|\phi\rangle\right) \;=\; \langle\psi|\phi\rangle \;=\; \sum_i a_i^* b_i \;\in\; \mathbb{C}.$$

The Riesz Representation Theorem

Theorem (Riesz): For every linear functional $f \in \mathscr{H}^*$, there exists a *unique* vector $|\psi_f\rangle \in \mathscr{H}$ such that

$$f(|\phi\rangle) = \langle \psi_f \mid \phi \rangle \quad \text{for all } |\phi\rangle \in \mathscr{H}.$$

- **Consequence:** there is a canonical *antilinear* isomorphism $\mathscr{H} \cong \mathscr{H}^*$:

  $$|\psi\rangle \longleftrightarrow \langle\psi|, \qquad \alpha\,|\psi\rangle \longleftrightarrow \alpha^*\,\langle\psi|.$$

  (Antilinear because the adjoint conjugates scalars.)
- **Significance:**
  - Every bra is the image of a unique ket — there is no "orphan" functional.
  - The inner product $\langle\phi|\psi\rangle$ is *literally* the action of the functional $\langle\phi|$ on the vector $|\psi\rangle$.
  - This is the rigorous foundation for Dirac's bracket notation.

Orthonormal Bases

A basis $\{|e_i\rangle\}$ for $\mathscr{H}$ is **orthonormal** if:

$$\langle e_i|e_j\rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad \text{(Kronecker delta).}$$

- **Expansion:** any state decomposes uniquely as

$$|\psi\rangle = \sum_i c_i \, |e_i\rangle \,, \qquad c_i = \langle e_i|\psi\rangle \,.$$

  The coefficient c_i is the *projection* (component) of $|\psi\rangle$ along $|e_i\rangle$.
- **Parseval identity:** for a normalized state,

$$\| \, |\psi\rangle \, \|^2 = \sum_i |c_i|^2 = 1,$$

  so the squared moduli $|c_i|^2$ are non-negative and sum to one — they will become the Born-rule probabilities in Lecture 3.

The Gram-Schmidt Process

Given any linearly independent set $\{|v_1\rangle, \dots, |v_n\rangle\}$, the **Gram-Schmidt process** produces an orthonormal set $\{|e_1\rangle, \dots, |e_n\rangle\}$ spanning the same subspace:

1. Set $|u_1\rangle = |v_1\rangle$ and normalize: $|e_1\rangle = \dfrac{|u_1\rangle}{\| \, |u_1\rangle \, \|}$.
2. For $k = 2, \dots, n$, subtract the already-found components:

$$|u_k\rangle = |v_k\rangle - \sum_{j=1}^{k-1} \langle e_j | v_k \rangle \, |e_j\rangle \,, \qquad |e_k\rangle = \frac{|u_k\rangle}{\| \, |u_k\rangle \, \|}.$$

- Each step removes all previously found directions from $|v_k\rangle$, leaving only the orthogonal remainder.
- **Implication:** every finite-dimensional inner product space possesses an orthonormal basis, and we can always choose to work in one.

The Outer Product

With both kets and bras in hand, we form the **outer product** $|\phi\rangle\langle\psi|$, which is a linear operator on $\mathscr{H}$:

$$\big(|\phi\rangle\langle\psi|\big)\,|\gamma\rangle = |\phi\rangle\ \langle\psi|\gamma\rangle = \langle\psi|\gamma\rangle\ |\phi\rangle\,.$$

- **Matrix form:** a column ($d \times 1$) times a row ($1 \times d$) yields a $d \times d$ matrix.
- **Rank-1 projector:** when $\|\,|\psi\rangle\,\| = 1$, the operator $P_\psi = |\psi\rangle\langle\psi|$ satisfies:
  - $P_\psi^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = P_\psi$ (idempotent).
  - $P_\psi^\dagger = \big(|\psi\rangle\langle\psi|\big)^\dagger = |\psi\rangle\langle\psi| = P_\psi$ (Hermitian).

  This answers the question: an orthogonal projector is Hermitian because $|\psi\rangle\langle\psi|$ is its own conjugate transpose.
- The complementary projector $Q = I - P_\psi$ projects onto the subspace orthogonal to $|\psi\rangle$.

The Completeness Relation (Resolution of the Identity)

Theorem: For any orthonormal basis $\{|e_i\rangle\}$ of $\mathscr{H}$:

$$\sum_i |e_i\rangle \langle e_i| = I.$$

Proof: Apply both sides to an arbitrary $|\psi\rangle = \sum_j c_j |e_j\rangle$:

$$\left(\sum_i |e_i\rangle \langle e_i|\right) |\psi\rangle = \sum_i |e_i\rangle \langle e_i|\psi\rangle = \sum_i c_i |e_i\rangle = |\psi\rangle . \checkmark$$

- **Operator representation:** inserting $I = \sum_i |e_i\rangle \langle e_i|$ on both sides of A,
$$A = \sum_{i,j} |e_i\rangle \underbrace{\langle e_i| A |e_j\rangle}_{A_{ij}} \langle e_j| = \sum_{i,j} A_{ij} \, |e_i\rangle \langle e_j| .$$
- **Inner product computation:** $\langle\phi|A|\psi\rangle = \sum_i \langle\phi|e_i\rangle \langle e_i| A |\psi\rangle$ (basis expansion / change of basis).
- **Born rule preview:** $P(\text{outcome } i) = |\langle e_i|\psi\rangle|^2 = \langle\psi| |e_i\rangle \langle e_i| |\psi\rangle$.

The Spectral Decomposition Theorem

Theorem: Every **normal** operator ($AA^\dagger = A^\dagger A$) on a finite-dimensional Hilbert space admits a spectral decomposition:

$$A = \sum_i \lambda_i \, |e_i\rangle \langle e_i| \, ,$$

where $\lambda_i \in \mathbb{C}$ are the eigenvalues and $\{|e_i\rangle\}$ is an orthonormal basis of eigenvectors.

- **Hermitian** operators ($A = A^\dagger$) are normal and have *real* eigenvalues — proved as follows:

$$\lambda \langle \psi | \psi \rangle = \langle \psi | A | \psi \rangle = \langle A\psi | \psi \rangle = \lambda^* \langle \psi | \psi \rangle \;\Rightarrow\; \lambda = \lambda^* \;\Rightarrow\; \lambda \in \mathbb{R}.$$

- **Unitary** operators ($U^\dagger U = I$) are normal and have eigenvalues on the *unit circle*: $|\lambda_i| = 1$.
- **Functions of operators:** for any function f,

$$f(A) = \sum_i f(\lambda_i) \, |e_i\rangle \langle e_i| \, .$$

For example, $e^{i\theta A} = \sum_i e^{i\theta\lambda_i} \, |e_i\rangle \langle e_i|$ (quantum time evolution).

Tensor Products and the Kronecker Product

To describe *composite* systems (multiple qubits), we need the **tensor product**.

- If system 1 is in $\mathscr{H}_1 \cong \mathbb{C}^m$ and system 2 is in $\mathscr{H}_2 \cong \mathbb{C}^n$, the joint Hilbert space is $\mathscr{H}_1 \otimes \mathscr{H}_2 \cong \mathbb{C}^{mn}$.
- A **product state**: $|\psi\rangle \otimes |\phi\rangle \equiv |\psi\phi\rangle$.
- **Kronecker product** (matrix representation): for A ($m \times m$) and B ($n \times n$),

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mm}B \end{bmatrix}, \quad \text{size } mn \times mn.$$

- **Two-qubit example:** $|0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \equiv |01\rangle$.

Summary of Lecture 2

- A **Hilbert space** $\mathscr{H}$ is a complete complex inner product space; the inner product is linear in the second argument, conjugate-linear in the first; it induces a norm $\|v\| = \sqrt{\langle v, v\rangle}$.
- **Kets** $|\psi\rangle$ are column vectors (states); the **adjoint** $A^\dagger$ of an operator is defined by $\langle Ah, k\rangle = \langle h, A^\dagger k\rangle$. Hermitian: $A = A^\dagger$; Unitary: $U^\dagger U = I$.
- **Bras** $\langle\psi| = |\psi\rangle^\dagger$ are elements of the dual space $\mathscr{H}^*$. The **Riesz theorem** guarantees the antilinear isomorphism $|\psi\rangle \leftrightarrow \langle\psi|$.
- Any orthonormal basis satisfies $\langle e_i|e_j\rangle = \delta_{ij}$ and the **completeness relation** $\sum_i |e_i\rangle\langle e_i| = I$; the Gram-Schmidt process constructs one from any linearly independent set.
- The **spectral theorem** decomposes any normal operator as $A = \sum_i \lambda_i |e_i\rangle\langle e_i|$, with real λ_i for Hermitian A.
- **Tensor products** build multi-qubit spaces: n qubits require $\mathbb{C}^{2^n}$.

Looking Ahead: Lecture 3

With the mathematical language in place, we are ready to state quantum mechanics axiomatically.

In Lecture 3 we will cover:

- **The four postulates** of quantum mechanics.
- **Measurement and observables:** projective measurements and the Born rule.
- **Pure vs. mixed states:** the density operator ρ.
- **Composite systems:** entanglement, separability, and Bell states.
- **Partial trace** and reduced density matrices.