Lecture 11: Quantum Noise and Quantum Operations

Axiomatic Framework, Kraus Representations, and Geometric Channels

Moody T. Chu

North Carolina State University

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Lecture Overview

1. Limitations of Closed-System Unitary Dynamics
2. The Global State Space Framework and Partial Trace
3. Axiomatic Formulation of Open Quantum Systems
4. The Operator-Sum Representation (Kraus Representation Theorem)
5. Unitary Freedom and Equivalence of Noise Maps
6. Discrete Qubit Noise: Bit-Flip, Phase-Flip, and Depolarizing Channels
7. Continuous Dissipative Noise: Amplitude and Phase Damping Models
8. Geometric Characterization of Noise via Bloch Ball Deformations

Limitations of Closed-System Unitary Dynamics

- **The postulate restriction:** Ideal quantum mechanics dictates that a closed system $\mathscr{H}_S$ transforms via a unitary matrix: $\rho \to U\rho U^\dagger$.
  - Isolation is an unattainable mathematical limit.
  - Laboratory systems interact with surrounding physical backgrounds (thermal reservoirs, stray fields).
- **The open reality:** Open systems inevitably interact with an external environment space $\mathscr{H}_E$.
  - The global state in $\mathscr{H}_S \otimes \mathscr{H}_E$ still evolves unitarily, but the system-only dynamics are not.
  - Open systems exhibit non-unitary traits: pure states transition into mixed ensembles, and quantum information (coherence) leaks out.
- **Goal:** Find a map $\mathcal{E} : \mathcal{L}(\mathscr{H}_S) \to \mathcal{L}(\mathscr{H}_S)$ that models state changes of the system entirely from its own perspective, by tracing out the environmental degrees of freedom:

$$\mathcal{E}(\rho_S) = \mathrm{Tr}_E \left[U \left(\rho_S \otimes |e_0\rangle \langle e_0| \right) U^\dagger \right].$$

Why Unitary Dynamics Fail

- When a qubit is not isolated,
  - It interacts with stray fields and thermal fluctuations, causing *decoherence* — the irreversible loss of quantum information.
  - Unitary evolution is information-preserving. Mathematically, no unitary U can map a pure state to a mixed state.
- If the qubit loses purity,
  - The global unitary U on (System + Environment) still applies, but a system-only unitary U_S is mathematically impossible.
  - We need $\mathcal{E}$ to characterize the phenomenon:
    - $\mathcal{E}$ represents mathematically the irreversible transition from quantum pure states to classical mixed states.
    - It is the mathematical projection of the global unitary evolution onto the system-only subspace.

Comparison: Unitary Dynamics vs. Quantum Maps

Feature	**Unitary (U)**	**Quantum Map ($\mathcal{E}$)**
System Type	Closed (Isolated)	Open (Interacting)
Information	Retained (Reversible)	Leaked (Irreversible)
Output State	Pure remains Pure	Pure often becomes Mixed
Dynamics	$\rho \to U\rho U^\dagger$	$\rho \to \sum_k E_k \rho E_k^\dagger$

- **The fundamental distinction:**
  - Unitary evolution preserves the total information of the system.
  - Quantum maps model the *statistical loss* of information that occurs when a system is coupled to an environment.

The Global State Space Framework

- Model the open system by constructing a composite, closed universe containing both the system ($\mathscr{H}_S$) and the environment ($\mathscr{H}_E$).
  - ◇ The combined state space is defined by the tensor product $\mathscr{H}_S \otimes \mathscr{H}_E$.
  - ◇ Assume the environment begins in a well-characterized, pure reference state $|e_0\rangle \in \mathscr{H}_E$.
- **Initial Product State Assumption:** Assume the joint state is initially completely separable:

$$\rho_{\text{global}}(0) = \rho_S \otimes |e_0\rangle \langle e_0| \quad \in \mathcal{L}(\mathscr{H}_S \otimes \mathscr{H}_E).$$

- The global system then evolves unitarily under U acting on $\mathscr{H}_S \otimes \mathscr{H}_E$, and the reduced state of the system is recovered by tracing out the environment:

$$\mathcal{E}(\rho_S) = \mathrm{Tr}_E \left[U \left(\rho_S \otimes |e_0\rangle \langle e_0| \right) U^\dagger \right].$$

The Partial Trace: Tracing Out the Environment

- Let $\{|e_k\rangle\}$ form an orthonormal basis for the d_E-dimensional environment space $\mathscr{H}_E$. Inserting the resolution of the identity $I_E = \sum_k |e_k\rangle\langle e_k|$ into the trace gives:

$$\mathcal{E}(\rho_S) = \sum_{k=0}^{d_E-1} (I_S \otimes \langle e_k|)\, U\, (\rho_S \otimes |e_0\rangle\langle e_0|)\, U^\dagger (I_S \otimes |e_k\rangle).$$

- Each term $(I_S \otimes \langle e_k|)(\cdots)(I_S \otimes |e_k\rangle)$ projects the global state onto the k-th environmental outcome.
- This operation maps a valid density matrix to another valid density matrix within $\mathscr{H}_S$, discarding the environmental degrees of freedom.
- **Key question:** Can we simplify this expression into a clean, operator-level formula? $\Longrightarrow$ Yes: this leads directly to the Kraus representation.

Dealing with an Unknown Environment

- Most of the time, we do not know the environment explicitly.
  - Instead, we seek an *axiomatic* approach that frees us from needing an explicit microscopic model of the environment.
  - The environment has carried away information and replaced it with statistical noise; the map $\mathcal{E}$ is irreversible.
  - Any abstract "black-box" map $\mathcal{E}$ satisfying three mathematical axioms is guaranteed to have originated from some real, physical interaction with an environment.
    - This guarantee is known as the **Stinespring Dilation Theorem**.
    - It ensures physical realizability without requiring explicit knowledge of the environment.

Axiomatic Definition of a Quantum Operation

- An open system transformation $\mathcal{E}$ is a valid quantum operation if it satisfies three mathematical properties:
  1. **Linearity:** $\mathcal{E}$ linearly preserves statistical mixtures:

$$\mathcal{E}\left(\sum_i p_i \rho_i\right) = \sum_i p_i \mathcal{E}(\rho_i), \quad \sum_i p_i = 1.$$

  2. **Trace Condition:** For deterministic (trace-preserving) channels, $\mathrm{Tr}(\mathcal{E}(\rho)) = \mathrm{Tr}(\rho) = 1$. For non-trace-preserving maps (post-measurement), $\mathrm{Tr}(\mathcal{E}(\rho)) \leq 1$.
  3. **Complete Positivity (CP):** For any auxiliary system $\mathscr{H}_R$ of any dimension, the extended map must remain positive semi-definite:

$$(I_R \otimes \mathcal{E})(\rho_{RS}) \geq 0 \quad \forall\, \rho_{RS} \in \mathcal{L}(\mathscr{H}_R \otimes \mathscr{H}_S).$$

Why Complete Positivity?

- A map $\mathcal{E}$ is *positive* if $\mathcal{E}(\rho) \geq 0$ whenever $\rho \geq 0$. But if the system is entangled with an auxiliary system, we need a stronger guarantee.
- Consider the transpose map $\mathcal{T}(\rho) = \rho^T$. Trivially, $\rho^T \geq 0$ when $\rho \geq 0$, so $\mathcal{T}$ is positive.
  - ◇ Suppose S is entangled with an auxiliary system R in the Bell state $|\psi^+\rangle_{RS} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
  - ◇ Then $(I_R \otimes \mathcal{T})(\rho_{RS}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, with eigenvalues $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\mathbf{\frac{1}{2}}\right\}$.
    - A negative eigenvalue of a density matrix is physically impossible (probabilities cannot be negative). Hence $\mathcal{T}$ is *not* completely positive.
- **Conclusion:** Simple positivity is insufficient. Complete positivity is the correct physical requirement for any map involving entangled states.

Derivation of the Kraus Representation

- Begin with the partial trace formulation:

$$\mathcal{E}(\rho_S) = \sum_{k=0}^{d_E-1} (I_S \otimes \langle e_k|)\, U\, (\rho_S \otimes |e_0\rangle \langle e_0|)\, U^\dagger (I_S \otimes |e_k\rangle).$$

- Use the factorization identity $\rho_S \otimes |e_0\rangle \langle e_0| = (I_S \otimes |e_0\rangle)\, \rho_S\, (I_S \otimes \langle e_0|)$ to separate the initial environment state:

$$\begin{aligned} I_S \otimes |e_0\rangle \langle e_0| &= (I_S \otimes |e_0\rangle)(I_S \otimes \langle e_0|), \\ (\rho_S \otimes I_E)(I_S \otimes |e_0\rangle) &= (I_S \otimes |e_0\rangle)\, \rho_S, \\ \rho_S \otimes |e_0\rangle \langle e_0| &= (I_S \otimes |e_0\rangle)\, \rho_S\, (I_S \otimes \langle e_0|). \end{aligned}$$

- Upon substitution:

$$\mathcal{E}(\rho_S) = \sum_{k=0}^{d_E-1} \underbrace{[(I_S \otimes \langle e_k|)\, U\, (I_S \otimes |e_0\rangle)]}_{E_k}\, \rho_S\, \underbrace{\left[(I_S \otimes \langle e_0|)\, U^\dagger (I_S \otimes |e_k\rangle)\right]}_{E_k^\dagger}.$$

Kraus Representation: Operator-Sum Form

- Define the **Kraus operator** associated with the k-th environmental outcome:

$$E_k := (I_S \otimes \langle e_k|)\, U\, (I_S \otimes |e_0\rangle).$$

  - ◇ E_k maps $\mathscr{H}_S \to \mathscr{H}_S$ and encodes the effect on the system when the environment transitions to $|e_k\rangle$.
  - ◇ How much information of U can be recovered from $\{E_k\}$?

- The **Kraus Representation** (Operator-Sum Representation):

$$\mathcal{E}(\rho) = \sum_{k=0}^{d_E-1} E_k \rho E_k^\dagger. \quad (1)$$

- This expresses the effect of the environment entirely in terms of system-level operators $\{E_k\}$, with no explicit reference to $\mathscr{H}_E$.

Completeness Relation

- The Kraus operators must satisfy the **completeness relation** $\sum_k E_k^\dagger E_k = I_S$, which encodes trace-preservation. To verify:

$$
\begin{aligned}
\sum_k E_k^\dagger E_k &= \sum_k \left[\left(I_S \otimes \langle e_k|\right) U \left(I_S \otimes |e_0\rangle\right)\right]^\dagger \left[\left(I_S \otimes \langle e_k|\right) U \left(I_S \otimes |e_0\rangle\right)\right] \\
&= \left(I_S \otimes \langle e_0|\right) U^\dagger \left(\sum_k I_S \otimes |e_k\rangle \langle e_k|\right) U \left(I_S \otimes |e_0\rangle\right) \\
&= \left(I_S \otimes \langle e_0|\right) U^\dagger \left(I_S \otimes I_E\right) U \left(I_S \otimes |e_0\rangle\right) \\
&= \left(I_S \otimes \langle e_0|\right) \left(I_{SE}\right) \left(I_S \otimes |e_0\rangle\right) = I_S \otimes \langle e_0|e_0\rangle = I_S.
\end{aligned}
$$

- **Significance:** Any map $\mathcal{E}$ satisfying the three axioms (Linearity, Trace Condition, CP) admits a Kraus decomposition. This bypasses the need to model the environment explicitly.

Non-Uniqueness and Unitary Freedom

- **Theorem (Unitary Freedom in Kraus Representation):** The Kraus operators $\{E_k\}$ for a quantum operation $\mathcal{E}$ are *not unique*.
  - ◇ Two sets of Kraus operators $\{E_k\}_{k=1}^{m}$ and $\{F_j\}_{j=1}^{m}$ generate the same quantum map $\mathcal{E}$ if and only if they are related by a unitary matrix $u = [u_{jk}]$:

$$F_j = \sum_k u_{jk} \, E_k .$$

  - ◇ **Linear algebraic insight:** This non-uniqueness corresponds exactly to making an alternate choice of orthonormal basis $\{|e_k\rangle\}$ in the environment space $\mathscr{H}_E$.
- Consequently, a single physical noise process can be analyzed using entirely different, mathematically equivalent sets of Kraus operators.
- We will see a concrete example at the end: the *phase flip* and *phase damping* channels are the same map in different Kraus bases.

Motivating Example: The Bit-Flip Channel

- Consider a qubit that accidentally flips (X gate) with probability p.
  - ◇ This probabilistic phenomenon cannot be written as a single unitary U_S.
- Define a map that captures the statistical average:

$$\mathcal{E}(\rho) = (1 - p)\rho + p\,X\rho X.$$

- If we start with the pure state $|0\rangle$, we end with a mixed state:

$$\rho_{out} = (1 - p)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + p\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - p & 0 \\ 0 & p \end{pmatrix}.$$

- Purity check: $\mathrm{Tr}(\rho_{out}^2) = (1 - p)^2 + p^2 = 1 - 2p(1 - p) < 1$ for $0 < p < 1$.
- The map $\mathcal{E}$ summarizes the noise without needing to know the state of the environment — precisely what the Kraus formalism provides.

A Preview of the Noise Channels

- We now systematically derive the Kraus representations for five standard noise models. For each channel, we follow the same three-step procedure:
  1. **Stinespring Dilation:** Write a global unitary U on $\mathscr{H}_S \otimes \mathscr{H}_E$ encoding the physical noise process.
  2. **Partial Trace:** Compute $\mathcal{E}(\rho_S) = \mathrm{Tr}_E[U(\rho_S \otimes |e_0\rangle\langle e_0|)U^\dagger]$ to obtain the Kraus operators E_k.
  3. **Bloch Ball Geometry:** Determine how the map deforms the Bloch sphere.
- The five channels:
  - **Discrete:** Bit-flip, Phase-flip, Depolarizing.
  - **Continuous:** Amplitude damping, Phase damping.

The Bit-Flip Channel

- **Physical action:** The qubit preserves its state with probability $1-p$, and undergoes a bit inversion (Pauli X) with probability p.
- **Stinespring dilation:**

$$U(|\psi\rangle \otimes |0\rangle_E) = \sqrt{1-p}\, |\psi\rangle |0\rangle_E + \sqrt{p}\, (X|\psi\rangle) |1\rangle_E .$$

  - ◇ When a flip occurs, the environment transitions to $|1\rangle_E$, entangling system with environment.
- **Kraus operators** (projecting onto $|0\rangle_E$ and $|1\rangle_E$):

$$E_0 = \sqrt{1-p}\, I = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad E_1 = \sqrt{p}\, X = \begin{bmatrix} 0 & \sqrt{p} \\ \sqrt{p} & 0 \end{bmatrix}.$$

- **Channel output:**

$$\mathcal{E}(\rho) = (1-p)\rho + p\, X\rho X = \begin{bmatrix} (1-p)\rho_{00} + p\rho_{11} & (1-2p)\rho_{01} \\ (1-2p)\rho_{10} & (1-p)\rho_{11} + p\rho_{00} \end{bmatrix}.$$

Geometric Deformation: Bit-Flip Channel

- Recall that any qubit density matrix is parametrized by the Bloch vector $\mathbf{r} = (x, y, z)^T$:

$$\rho = \frac{1}{2}(I + xX + yY + zZ) = \begin{bmatrix} \frac{1+z}{2} & \frac{x-iy}{2} \\ \frac{x+iy}{2} & \frac{1-z}{2} \end{bmatrix}.$$

- Using $X\sigma_\mu X = +\sigma_\mu$ for $\mu = 0, X$ and $X\sigma_\mu X = -\sigma_\mu$ for $\mu = Y, Z$, the channel gives:

$$\mathcal{E}(\rho) = \frac{1}{2}(I + x\,X + (1-2p)y\,Y + (1-2p)z\,Z)\,,$$

so the Bloch coordinates transform as:

$$x' = x, \quad y' = (1-2p)\,y, \quad z' = (1-2p)\,z.$$

- **Geometric deformation:** The Bloch sphere is compressed into an ellipsoid elongated along the x-axis. The x-direction (the eigenbasis of X) is unaffected; all other directions contract.

The Phase-Flip Channel

- **Physical action:** The phase coherence of superpositions is destroyed with probability p, while the computational basis states $|0\rangle, |1\rangle$ are left untouched.
- **Stinespring dilation:**

$$
\begin{aligned}
U(|0\rangle_S \otimes |e_0\rangle_E) &= |0\rangle_S \otimes |e_0\rangle_E\,, \\
U(|1\rangle_S \otimes |e_0\rangle_E) &= |1\rangle_S \otimes \left(\sqrt{1-p}\,|e_0\rangle_E + \sqrt{p}\,|e_1\rangle_E\right).
\end{aligned}
$$

  - ◇ The state $|0\rangle_S$ is unaffected. Only $|1\rangle_S$ becomes entangled with the environment.
  - ◇ $\sqrt{p}$ controls how strongly the environment couples to $|1\rangle_S$.
- **Kraus operators** (projecting onto $|e_0\rangle_E$ and $|e_1\rangle_E$):

$$
E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{p} \end{bmatrix}.
$$

The Phase-Flip Channel: Output and Geometry

- **Channel output:**

$$\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger = \begin{bmatrix} \rho_{00} & \sqrt{1-p}\,\rho_{01} \\ \sqrt{1-p}\,\rho_{10} & \rho_{11} \end{bmatrix}.$$

  - ◇ Populations ρ_{00}, ρ_{11} are preserved; coherences ρ_{01}, ρ_{10} are damped by $\sqrt{1-p}$.
- The channel can equivalently be written as $\mathcal{E}(\rho) = (1-\gamma)\rho + \gamma\, Z\rho Z$, where $\gamma = \frac{1-\sqrt{1-p}}{2}$.
  - ◇ Check: $(1-\gamma)\rho + \gamma Z\rho Z$ replaces $\rho_{01} \to (1-2\gamma)\rho_{01} = \sqrt{1-p}\,\rho_{01}$. ✓
  - ◇ This alternative Kraus set $\{E_0' = \sqrt{1-\gamma}\, I,\ E_1' = \sqrt{\gamma}\, Z\}$ is related to the original by the Unitary Freedom theorem.
- **Bloch vector:**

$$x' = \sqrt{1-p}\, x, \quad y' = \sqrt{1-p}\, y, \quad z' = z.$$

  - ◇ The sphere contracts uniformly in the x-y plane; the z-axis is completely unaffected.

The Phase-Flip Channel: Where Is the Flip?

- Consider $|+\rangle_S = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ under the channel. The global output splits into two branches:

$$|\Psi_{\text{out}}\rangle = \underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle_S + \sqrt{1-p}\,|1\rangle_S\right) \otimes |e_0\rangle_E}_{\text{Branch 0}} + \underbrace{\frac{\sqrt{p}}{\sqrt{2}}\,|1\rangle_S \otimes |e_1\rangle_E}_{\text{Branch 1}}.$$

  - ◇ Neither branch contains a literal negative sign. So where is the phase flip?

- Decompose Branch 0 in the $\{|+\rangle, |-\rangle\}$ basis, using $|0\rangle = \frac{|+\rangle+|-\rangle}{\sqrt{2}}$ and $|1\rangle = \frac{|+\rangle-|-\rangle}{\sqrt{2}}$:

$$\frac{1}{\sqrt{2}}\left(|0\rangle_S + \sqrt{1-p}\,|1\rangle_S\right) = \frac{1+\sqrt{1-p}}{2}\,|+\rangle + \frac{1-\sqrt{1-p}}{2}\,|-\rangle.$$

  - ◇ Branch 0 is a superposition of $|+\rangle$ (no error) and $|-\rangle$ (phase flip), with a literal relative minus sign between them. **The flip is hidden inside Branch 0.**

Why Is This Phase-Flip Noise Significant?

- In the $\{|+\rangle, |-\rangle\}$ basis, $|\Psi\rangle_{\text{out}}$ can be written as:

$$|\Psi\rangle_{\text{out}} = |+\rangle \otimes \underbrace{\left(\tfrac{1+\sqrt{1-p}}{2}\, |e_0\rangle_E + \tfrac{\sqrt{p}}{2}\, |e_1\rangle_E\right)}_{|\Phi_{\text{safe}}\rangle_E}$$
$$+\, |-\rangle \otimes \underbrace{\left(\tfrac{1-\sqrt{1-p}}{2}\, |e_0\rangle_E - \tfrac{\sqrt{p}}{2}\, |e_1\rangle_E\right)}_{|\Phi_{\text{flipped}}\rangle_E}.$$

  - ◇ The probability of finding the qubit in $|+\rangle$ is $\| \, |\Phi_{\text{safe}}\rangle_E \|^2 = 1 - \gamma$.
  - ◇ The probability of finding the qubit in $|-\rangle$ is $\| \, |\Phi_{\text{flipped}}\rangle_E \|^2 = \gamma$.
  - ◇ Both processes produce the same density matrix — the physical dampening is *indistinguishable* from a random sign-inversion error.
- **Implication for error correction:** Correcting phase-flip noise requires measuring in the $\{|+\rangle, |-\rangle\}$ basis, not the computational basis.

The Depolarizing Channel

- **Physical action:** The qubit remains unaffected with probability $1-p$, and is completely randomized into the maximally mixed state $I/2$ with probability p.
- To model isotropic (direction-independent) noise, the environment needs dimension at least 5: one "safe" state plus four states $|e_{ij}\rangle_E$ ($i,j \in \{0,1\}$) that track transitions $|j\rangle_S \to |i\rangle_S$.
- **Stinespring dilation:**

$$U = \sqrt{1-p}\, I_S \otimes |e_{\text{safe}}\rangle \langle e_0|_E + \sqrt{\frac{p}{2}} \sum_{i,j=0}^{1} |i\rangle_S \langle j|_S \otimes |e_{ij}\rangle \langle e_0|_E + U_*,$$

where U_* acts on the remaining environmental subspaces to ensure global unitarity (it never triggers since the environment starts in $|e_0\rangle_E$).

The Depolarizing Channel: Kraus Operators

- Projecting onto each environment basis state gives the Kraus operators:

$$E_{\text{safe}} = \sqrt{1-p}\, I_S, \quad E_{ij} = \sqrt{\tfrac{p}{2}}\, |i\rangle_S \langle j|_S\,, \quad i,j \in \{0,1\}.$$

- The channel output:

$$\begin{aligned}\mathcal{E}(\rho_S) &= (1-p)\rho_S + \frac{p}{2}\sum_{i,j} |i\rangle \underbrace{\langle j|\,\rho_S\,|j\rangle}_{\rho_{jj}} \langle i| \\ &= (1-p)\rho_S + \frac{p}{2}\underbrace{\left(\sum_j \rho_{jj}\right)}_{=1}\underbrace{\left(\sum_i |i\rangle\langle i|\right)}_{=I_S} = (1-p)\rho_S + p\,\frac{I_S}{2}.\end{aligned}$$

- Using the **Pauli twirl identity**: for any ρ with $\mathrm{Tr}(\rho)=1$,

$$\rho + X\rho X + Y\rho Y + Z\rho Z = 2\,\mathrm{Tr}(\rho)\, I = 2I,$$

so $\frac{I}{2} = \frac{1}{4}(\rho + X\rho X + Y\rho Y + Z\rho Z)$.

The Depolarizing Channel: Pauli Kraus Form and Geometry

- Substituting the Pauli twirl identity into $\mathcal{E}(\rho) = (1-p)\rho + p\,\frac{I}{2}$ reveals the isotropic structure:

$$\mathcal{E}(\rho) = \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}\left(X\rho X + Y\rho Y + Z\rho Z\right).$$

- This gives the standard Pauli-basis Kraus set:

$$E_0 = \sqrt{1 - \tfrac{3p}{4}}\, I, \quad E_1 = \sqrt{\tfrac{p}{4}}\, X, \quad E_2 = \sqrt{\tfrac{p}{4}}\, Y, \quad E_3 = \sqrt{\tfrac{p}{4}}\, Z.$$

- **Bloch vector:** Using $\sigma_\mu \sigma_\nu \sigma_\mu = -\sigma_\nu$ for $\mu \neq \nu$ (Pauli anticommutation):

$$x' = (1-p)\,x, \quad y' = (1-p)\,y, \quad z' = (1-p)\,z.$$

- **Geometric deformation:** Every Bloch vector shrinks isotropically toward the origin. As $p \to 1$, the entire sphere collapses to the single point $\mathbf{r} = 0$ (maximum entropy).

Amplitude Damping: Continuous Dissipative Noise

- **Physical action:** Models non-equilibrium energy loss — a physical atom emitting a photon into a cold reservoir, relaxing from the excited state $|1\rangle_S$ to the ground state $|0\rangle_S$ with decay parameter $\gamma \in [0, 1]$.
- **State space conventions:**
  - ◇ In $\mathscr{H}_S$: $|1\rangle_S$ is the excited (higher energy) state; $|0\rangle_S$ is the ground (lowest energy) state.
  - ◇ In $\mathscr{H}_E$: $|e_0\rangle_E$ is the vacuum (zero photons, cold reservoir); $|e_1\rangle_E$ is the one-photon state (environment has absorbed one quantum of energy).
- **Stinespring dilation** (environment starts cold in $|e_0\rangle_E$):

$$U\big(\,|0\rangle_S \otimes |e_0\rangle_E\,\big) = |0\rangle_S \otimes |e_0\rangle_E \quad \text{(ground state is stable)},$$
$$U\big(\,|1\rangle_S \otimes |e_0\rangle_E\,\big) = \sqrt{1-\gamma}\;|1\rangle_S \otimes |e_0\rangle_E + \sqrt{\gamma}\;|0\rangle_S \otimes |e_1\rangle_E\,.$$

  - ◇ With amplitude $\sqrt{1-\gamma}$: the atom retains its excitation (no emission).
  - ◇ With amplitude $\sqrt{\gamma}$: the atom decays ($|1\rangle_S \to |0\rangle_S$) and the environment absorbs one photon ($|e_0\rangle_E \to |e_1\rangle_E$).

Amplitude Damping: Kraus Operators and Channel Output

- **Kraus operators** (projecting onto environment basis states $|e_0\rangle_E$ and $|e_1\rangle_E$):

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}.$$

  - ◇ E_0: no emission; the atom either stays in $|0\rangle$ or remains in $|1\rangle$ with reduced amplitude.
  - ◇ E_1: spontaneous emission; the atom decays $|1\rangle_S \to |0\rangle_S$.
  - ◇ Completeness check: $E_0^\dagger E_0 + E_1^\dagger E_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1-\gamma \end{smallmatrix}\right] + \left[\begin{smallmatrix} 0 & 0 \\ 0 & \gamma \end{smallmatrix}\right] = I$. ✓

- **Channel output:**

$$\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger = \begin{bmatrix} \rho_{00} + \gamma \rho_{11} & \sqrt{1-\gamma}\,\rho_{01} \\ \sqrt{1-\gamma}\,\rho_{10} & (1-\gamma)\rho_{11} \end{bmatrix}.$$

  - ◇ The excited-state population ρ_{11} decays; the ground-state population ρ_{00} grows correspondingly.
  - ◇ Coherences are damped by $\sqrt{1-\gamma}$ (not $1-\gamma$; compare with phase damping).

Amplitude Damping: Bloch Deformation

- Converting the channel output to Bloch coordinates using $\rho_{00} = \frac{1+z}{2}$, $\rho_{11} = \frac{1-z}{2}$, $\rho_{01} = \frac{x-iy}{2}$:

$$x' = \sqrt{1-\gamma}\,x, \quad y' = \sqrt{1-\gamma}\,y, \quad z' = \gamma + (1-\gamma)z.$$

- As an affine map (linear transform plus a shift vector):

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \gamma \end{bmatrix}.$$

- **Geometric insight:** The Bloch sphere contracts asymmetrically *and* its center shifts upward along the z-axis toward the north pole $\mathbf{r} = (0,0,1)^T$, which represents the pure ground state $|0\rangle_S$. As $\gamma \to 1$, all states converge to $|0\rangle$.

The Phase Damping Channel

- **Physical action:** Models pure decoherence — the loss of phase information and quantum synchronization *without* any energy transfer.
- In quantum information, phase damping:
  - Does *not* change the populations (diagonal entries ρ_{00}, ρ_{11}).
  - Exponentially damps the quantum coherences (off-diagonal entries ρ_{01}, ρ_{10}).
- **Stinespring dilation:**

$$U\big(\,|0\rangle_S \otimes |e_0\rangle_E\,\big) = |0\rangle_S \otimes |e_0\rangle_E\,,$$
$$U\big(\,|1\rangle_S \otimes |e_0\rangle_E\,\big) = \sqrt{1-\gamma}\;|1\rangle_S \otimes |e_0\rangle_E + \sqrt{\gamma}\;|1\rangle_S \otimes |e_1\rangle_E\,.$$

  - Neither $|0\rangle_S$ nor $|1\rangle_S$ transitions to the other state, so populations are preserved.
  - The state $|1\rangle_S$ becomes *entangled* with the environment, destroying its phase coherence.

Phase Damping: Kraus Operators and Channel Output

- **Kraus operators:**

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix}.$$

- **Channel output:**

$$\mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger = \begin{bmatrix} \rho_{00} & \sqrt{1-\gamma}\,\rho_{01} \\ \sqrt{1-\gamma}\,\rho_{10} & \rho_{11} \end{bmatrix}.$$

  - ◇ Diagonal entries (populations) are preserved. Off-diagonal coherences are multiplied by $\sqrt{1-\gamma} < 1$.

- **Global unitary** in the basis $\{|0\rangle|e_0\rangle, |1\rangle|e_0\rangle, |0\rangle|e_1\rangle, |1\rangle|e_1\rangle\}$:

$$U = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & -\sqrt{\gamma} \\ \hline 0 & 0 & 1 & 0 \\ 0 & \sqrt{\gamma} & 0 & \sqrt{1-\gamma} \end{array}\right].$$

Phase Damping: Bloch Sphere Geometry

- **Bloch vector transformation:**

$$x' = \sqrt{1-\gamma}\, x, \quad y' = \sqrt{1-\gamma}\, y, \quad z' = z.$$

  - ◇ The transverse (x-y plane) components shrink; the z-axis is completely unchanged.
  - ◇ As $\gamma \to 1$, the Bloch sphere collapses onto a line segment along the z-axis — a classical mixture of $|0\rangle$ and $|1\rangle$ with no quantum coherence.
- **Contrast with amplitude damping:**
  - ◇ Amplitude damping shifts the center of the sphere toward $|0\rangle$ (energy loss to environment).
  - ◇ Phase damping preserves the center and populations but destroys the off-diagonal quantum correlations (information loss without energy exchange).
- **Physical significance:** Phase damping is the dominant decoherence mechanism in many systems (e.g., NMR T_2 relaxation, random laser phase kicks). It sets a fundamental timescale for how long a qubit retains its quantum information.

Channel Equivalence: Phase Damping = Phase Flip

- The phase damping ($\{E_k\}$) and phase-flip ($\{F_j\}$) channels are algebraically equivalent.
- **Phase-flip Kraus operators** (with parameter p):

$$F_0 = \sqrt{1-p}\, I, \quad F_1 = \sqrt{p}\, Z.$$

- By the Unitary Freedom theorem, there exists a unitary u such that $F_j = \sum_k u_{jk} E_k$.
- **Constructing the equivalence:** Set $p \equiv \frac{1-\sqrt{1-\gamma}}{2}$, which implies $\sqrt{\gamma} = 2\sqrt{p}\sqrt{1-p}$. Then:

$$u = \begin{bmatrix} \sqrt{1-p} & \sqrt{p} \\ \sqrt{p} & -\sqrt{1-p} \end{bmatrix},$$

which is a real, symmetric reflection matrix satisfying $u^\dagger u = I$.
- **Conclusion:** Phase damping and phase flipping are simply different Kraus-basis representations of the *same* quantum operation. They are physically identical and produce the same density matrix evolution. ■

Summary of This Lecture, I

- **Framework.** Open systems evolve via

$$\mathcal{E}(\rho_S) = \mathrm{Tr}_E[U(\rho_S \otimes |e_0\rangle\langle e_0|)U^\dagger].$$

This is non-unitary and irreversible — the mathematical signature of quantum noise.

- **Kraus Representation Theorem.** A map $\mathcal{E}$ is a valid quantum operation if and only if

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger,$$

with

$$\sum_k E_k^\dagger E_k = I.$$

The three axioms (Linearity, Trace Condition, Complete Positivity) are both necessary and sufficient.

Summary of This Lecture, II

- **Unitary Freedom.** Two Kraus sets $\{E_k\}$ and $\{F_j\}$ describe the same channel iff they are related by a unitary mixing matrix u:

$$F_j = \sum_k u_{jk} E_k.$$

- **Five Standard Channels and Their Bloch Deformations:**
  - ◇ *Bit-flip*: ellipsoid elongated along the x-axis (x-direction preserved).
  - ◇ *Phase-flip*: x-y coherences damped by $\sqrt{1-p}$; z-axis untouched.
  - ◇ *Depolarizing*: isotropic contraction toward the maximally mixed origin.
  - ◇ *Amplitude damping*: contraction toward $|0\rangle$ (energy loss to environment).
  - ◇ *Phase damping*: collapse onto z-axis; decoherence without energy loss.

Look Ahead: Lecture 12

- We have characterized *how noise acts* on a state ρ. The natural next question is: **how do we quantify how far apart two states are, or how much a channel has degraded the information it carries?**
- **Lecture 12: Distance Measures for Quantum Information**
  - **Classical foundations.** Trace distance $D(p,q) = \frac{1}{2}\sum_x |p_x - q_x|$ and classical fidelity $F(p,q) = \sum_x \sqrt{p_x q_x}$ as the starting point.
  - **Quantum trace distance.** $D(\rho,\sigma) = \frac{1}{2}\operatorname{Tr}|\rho - \sigma|$: a metric on state space, equal to half the Euclidean Bloch distance for qubits, and *contractive* under all quantum operations — noise can never increase distinguishability.
  - **Quantum fidelity and Uhlmann's Theorem.** $F(\rho,\sigma) = \operatorname{Tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$ is the maximum purification overlap: $F(\rho,\sigma) = \max_{|\phi\rangle} |\langle\psi|\phi|\psi|\phi\rangle|$.
  - **Fuchs–van de Graaf inequalities.** Linking the two measures: $1 - F \leq D \leq \sqrt{1 - F^2}$.
  - **Entanglement fidelity.** $F(\rho,\mathcal{E}) = \sum_k |\operatorname{Tr}(\rho E_k)|^2$: a *dynamic* measure of how faithfully a channel $\mathcal{E}$ preserves quantum correlations with the outside world.